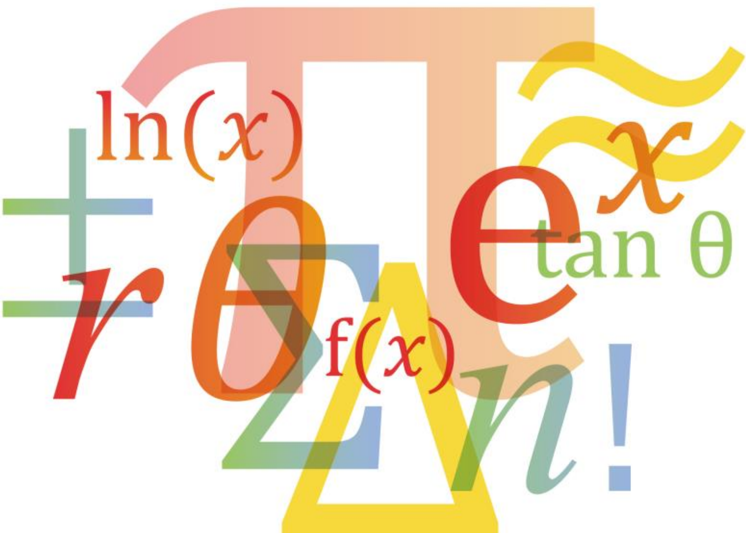
Cambridge Assessment International Education

Specimen Paper Answers – Paper 3

Cambridge International AS & A Level
Mathematics 9709

For examination from 2020



Version 2

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Contents

Introduction

The main aim of this booklet is to exemplify standards for those teaching Cambridge International AS & A Level Mathematics 9709, and to show examples of model answers to the 2020 Specimen Paper 3. Paper 3 assesses the syllabus content for Pure Mathematics 3. We have provided answers for each question in the specimen paper, along with examiner comments explaining where and why marks were awarded. Candidates need to demonstrate the appropriate techniques, as well as applying their knowledge when solving problems.

Individual examination questions may involve ideas and methods from more than one section of the syllabus content for that component. The main focus of examination questions will be the AS & A Level Mathematics subject content. However, candidates may need to make use of prior knowledge and mathematical techniques from previous study, as listed in the introduction to section 3 of the syllabus.

There are 9 to 11 structured questions in Paper 3; candidates must answer **all** questions. Questions are of varied lengths and often contain several parts, labelled (a), (b), (c), which may have sub-parts (i), (ii), (iii), as needed. Some questions might require candidates to sketch graphs or diagrams, or draw accurate graphs.

Candidates are expected to answer directly on the question paper. All working should be shown neatly and clearly in the spaces provided for each question. New questions often start on a fresh page, so more answer space may be provided than is needed. If additional space is required, candidates should use the lined page at the end of the question paper, where the question number or numbers must be clearly shown.

The mark schemes for the Specimen Papers are available to download from the School Support Hub at www.cambridgeinternational.org/support

2020 Specimen Mark Scheme 3

Past exam resources and other teacher support materials are available on the School Support Hub (www.cambridgeinternational.org/support).

Assessment overview

There are three routes for Cambridge International AS & A Level Mathematics. Candidates may combine components as shown below.

Route 1 **AS Level only** (Candidates take the AS components in the same series)	**Paper 1** **Pure Mathematics 1**	**Paper 2** **Pure Mathematics 2**	**Paper 3** **Pure Mathematics 3**	**Paper 4** **Mechanics**	**Paper 5** **Probability & Statistics 1**	**Paper 6** **Probability & Statistics 2**
Either	✓		Not available for AS Level		✓	Not available for AS Level
Or	✓			✓		
Or Note this option in Route 1 cannot count towards A Level	✓	✓				

Route 2 **A Level (staged over two years)**	**Paper 1** **Pure Mathematics 1**	**Paper 2** **Pure Mathematics 2**	**Paper 3** **Pure Mathematics 3**	**Paper 4** **Mechanics**	**Paper 5** **Probability & Statistics 1**	**Paper 6** **Probability & Statistics 2**
Either Year 1 AS Level	✓	Not available for A Level		✓		
Year 2 Complete the A Level			✓		✓	
Or Year 1 AS Level	✓				✓	
Year 2 Complete the A Level			✓			✓
Or Year 1 AS Level	✓				✓	
Year 2 Complete the A Level			✓	✓		

Route 3 **A Level** (Candidates take the A Level components in the same series)	**Paper 1** **Pure Mathematics 1**	**Paper 2** **Pure Mathematics 2**	**Paper 3** **Pure Mathematics 3**	**Paper 4** **Mechanics**	**Paper 5** **Probability & Statistics 1**	**Paper 6** **Probability & Statistics 2**
Either	✓	Not available for A Level	✓	✓	✓	
Or	✓		✓		✓	✓

Paper 3 – Pure Mathematics 3

- Written examination, 1 hour 50 minutes, 75 marks
- 9 to 11 structured questions based on the Pure Mathematics 3 subject content
- Candidates answer all questions
- Externally assessed by Cambridge International
- 30% of the A Level only

Compulsory for A Level.

Assessment objectives

The assessment objectives (AOs) are the same for all papers:

AO1 Knowledge and understanding

- Show understanding of relevant mathematical concepts, terminology and notation
- Recall accurately and use appropriate mathematical manipulative techniques

AO2 Application and communication

- Recognise the appropriate mathematical procedure for a given situation
- Apply appropriate combinations of mathematical skills and techniques in solving problems
- Present relevant mathematical work, and communicate corresponding conclusions, in a clear and logical way

Weightings for assessment objectives

The approximate weightings (± 5%) allocated to each of the AOs are summarised below.

Assessment objectives as an approximate percentage of each component

Assessment objective	Weighting in components %					
	Paper 1	Paper 2	Paper 3	Paper 4	Paper 5	Paper 6
AO1 Knowledge and understanding	55	55	45	55	55	55
AO2 Application and communication	45	45	55	45	45	45

Assessment objectives as an approximate percentage of each qualification

Assessment objective	Weighting in AS Level %	Weighting in A Level %
AO1 Knowledge and understanding	55	52
AO2 Application and communication	45	48

Question 1

1 Find the set of values of x for which $3(2^{3x+1}) < 8$. Give your answer in a simplified exact form. [3]

$$3 \times 2^{3x+1} < 8 \Rightarrow 6 \times 8^x < 8$$

$$8^{x-1} < \frac{1}{6}$$

$$x - 1 < \frac{\ln \frac{1}{6}}{\ln 8}$$

$$x < 1 + \frac{\ln \frac{1}{6}}{\ln 8} = 1 - \frac{\ln 6}{\ln 8}$$

Examiner comment

The approach used here involves rearranging and simplifying before considering logarithms, leading to the first M1A1 in line three. Either of the forms on the final line would be acceptable for the second A1. Note that the final A1 is for a correct **final** answer. A candidate who went on to apply incorrect rules and simplified this to $1 - \frac{\ln 3}{\ln 4}$ would **not** score the final A1.

Question 2

2 **(a)** Expand $(1+3x)^{-\frac{1}{3}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [3]

$$(1+3x)^{-\frac{1}{3}} = 1 + \left(-\frac{1}{3}\right)3x + \frac{\left(-\frac{1}{3}\right)\times\left(-\frac{4}{3}\right)}{2}(3x)^2$$

$$= 1 - x + 2x^2$$

Examiner comment

Many candidates do not show the unsimplified expansion. It is worth them taking the time to set this out correctly because it can prevent errors with the signs and errors when simplifying the fractions. A common cause of problems is the omission of brackets – a candidate who writes down what looks like $3x^2$ but clearly uses $(3x)^2 = 9x^2$ will be able to score full marks, but not squaring the 3 is a very common error.

(b) State the set of values of x for which the expansion is valid. [1]

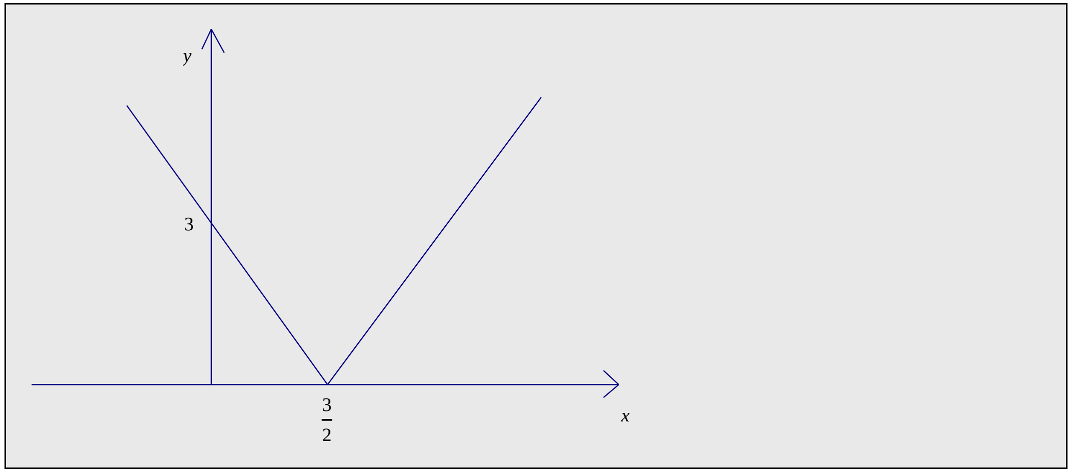
$$|3x| < 1 \quad \Rightarrow \quad |x| < \frac{1}{3}$$

Examiner comment

It is important to answer the question – the question wants a set of values for x, not for $3x$, so the first term alone would score B0. Any valid style for describing the interval is accepted.

Question 3

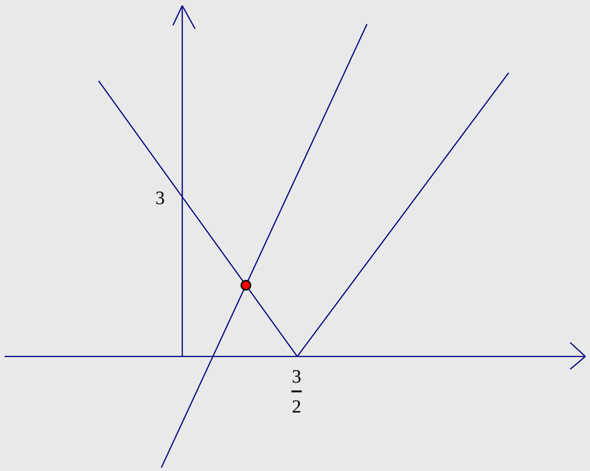
3 (a) Sketch the graph of $y = |2x - 3|$. [1]



Examiner comment

There is only one mark available for this sketch. The straight lines need to look straight, and the sketch needs to be symmetrical. Key values shown on the axes confirm that the sketch is correct.

(b) Solve the inequality $3x - 1 > |2x - 3|$. [3]



$3 - 2x = 3x - 1 \Rightarrow 5x = 4,\ x = \frac{4}{5}$

$\Rightarrow x > \frac{4}{5}$

Examiner comment

If the candidate finds it helpful, they could add an extra line to the sketch in part **(a)** to help them answer **(b)** without affecting their mark for **(a)**, so long as it is clearly an attempt to sketch $y = 3x - 1$. Drawing the extra line will help to identify where to look for the point of intersection, and also shows which side of the point you need to be to satisfy the inequality.

The most common approach to this type of question in the past has been for candidates to square both sides of the inequality to form a quadratic in x. This method is still valid, but squaring can introduce a solution that is incorrect. Note the new wording on the front page of this paper: '*You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator*.' A candidate who simply writes down the value $\frac{4}{5}$, without showing that it comes from correct working, will **not** gain credit.

Question 4

4 The parametric equations of a curve are

$$x = e^{2t-3}, \; y = 4\ln t,$$

where $t > 0$. When $t = a$ the gradient of the curve is 2.

(a) Show that a satisfies the equation $a = \frac{1}{2}(3 - \ln a)$. [4]

$x = e^{2t-3}$ $\qquad$ $y = 4\ln t$

$\frac{dx}{dt} = 2e^{2t-3}$ $\qquad$ $\frac{dy}{dt} = \frac{4}{t}$

$\Rightarrow \frac{dy}{dx} = \frac{\frac{4}{t}}{2e^{2t-3}} = \frac{2}{te^{2t-3}}$

When $t = a$, $2 = \frac{2}{ae^{2a-3}}$, $ae^{2a-3} = 1$

$\Rightarrow 0 = \ln a + (2a-3)$, $a = \frac{3-\ln a}{2}$

Examiner comment

Candidates do not need to quote the chain rule $\left(\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}\right)$, but if they don't state it and then make an error in applying it, they might not score any marks after the first B1. In a question like this, where the answer is given, it is particularly important to show sufficient working to support the answer – candidates should not miss out any steps. The final answer must be the answer given in the question, if candidates have not substituted *a* in place of *t* in the final equation, then they can only be awarded a maximum of three marks.

(b) Verify by calculation that this equation has a root between 1 and 2. [2]

Using $f(a) = 2a + \ln a - 3$, $f(1) = -1 < 0$

$f(2) = 1.69 > 0$

Change of sign implies root between 1 and 2.

Examiner comment

The key step is to choose an appropriate function to evaluate. If candidates are applying the sign change rule then they need to rearrange the equation to the form $f(a) = 0$. There is more than one correct way to do this; the examiners will be aware of the values obtained by using the most common forms for $f(a)$.

Many candidates lose marks here because they do not say what they are doing and they also make errors in the calculations. So long as the values obtained are stated to sufficient significant figures to imply that the process has been completed correctly, it is not necessary to give the answers to any particular degree of accuracy. If a sign change is not observed, candidates should go back and check that the function is correct and that they have not made a calculation error. It is not sufficient just to find two correct values – a brief comment is also required, confirming the change of sign.

(c) Use the iterative formula $a_{n+1} = \frac{1}{2}(3 - \ln a_n)$ to calculate a correct to 2 decimal places, showing the result of each iteration to 4 decimal places. [3]

$a_0 = 1$

$a_1 = 1.5$

$a_2 = 1.2973$

$a_3 = 1.3699$

$a_4 = 1.3426$

$\left.\begin{array}{l} a_5 = 1.3527 \\ a_6 = 1.3490 \end{array}\right\} \quad \alpha = 1.35$

$a_7 = 1.3503$

Examiner comment

The first step is to make an appropriate choice of value for a_0. Following part **(b)**, it is sensible to choose a_0 so that $1 \le a_0 \le 2$. The question requires the result of each iteration to be given to 4 decimal places. Many candidates lose marks in this type of question because they do not give their answers to the required accuracy. Note that in this particular example, the value of a_1 is exact, so candidates would not be penalised for not giving the value to 4 decimal places. Candidates need to continue their working until they obtain two values that both give the same value when rounded to two decimal places. If they do not give a recognisable sequence of iterations then no mark will be awarded for the final answer.

In each examination series, there are always a few candidates who do not appear to understand how to work through the iterative method. The simplest approach is to choose the starting value and enter this on the calculator, so for the above example they would key in “1 =”. They then need to key in the formula $\frac{1}{2}(3 - \ln(\text{"ans"}))$ and record the value shown each time they press “=”.

Question 5

5 (a) Show that $\frac{\mathrm{d}}{\mathrm{d}x}(x - \tan^{-1} x) = \frac{x^2}{1 + x^2}$. [2]

$$\frac{d}{dx}\left(x - \tan^{-1} x\right) = 1 - \frac{1}{1 + x^2}$$

$$= \frac{1 + x^2 - 1}{1 + x^2} = \frac{x^2}{1 + x^2}$$

Examiner comment

This question has a given answer, so the response needs to show sufficient working – candidates must show the derivative of each term separately before attempting to simplify the answer. The differentiation of $\tan^{-1}x$ (and the integration of $\frac{1}{1+x^2}$) is a new addition to the 2020 specification. The marks available in this question are an indication that all that is needed here is the use of the result – for this question it is not necessary to demonstrate how to obtain the derivative of $\tan^{-1}x$.

(b) Show that $\int_0^{\sqrt{3}} x\tan^{-1}x\,dx = \frac{2}{3}\pi - \frac{1}{2}\sqrt{3}$. [5]

$$\int x\tan^{-1}x\,dx = \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int\frac{x^2}{1+x^2}dx$$

$$= \left[\frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x\right]_0^{\sqrt{3}}$$

$$= \frac{3}{2}\tan^{-1}\sqrt{3} - \frac{\sqrt{3}}{2} + \frac{1}{2}\tan^{-1}\sqrt{3} - 0 = \frac{\pi}{3}\left(\frac{3}{2}+\frac{1}{2}\right) - \frac{\sqrt{3}}{2}$$

$$= \frac{2}{3}\pi - \frac{1}{2}\sqrt{3}$$

Examiner comment

The first three marks here are for the correct application of the method for integration by parts – the limits do not need to be seen until the next stage. The value of the integral is given in an exact form, so all working must be exact. If there is any evidence of the use of decimal approximations in the working then all subsequent accuracy marks will be lost.

The second method mark is only available after a complete attempt to apply integration by parts. For this method mark, there needs to be evidence that both limits have been used – the solution needs to show that the values obtained when substituting zero have been considered, hence the inclusion of " – 0" in the penultimate line. The final answer is given in the question, so full and clear working must be shown.

Question 6

6 The complex numbers 1 + 3i and 4 + 2i are denoted by *u* and *v* respectively.

(a) Find $\frac{u}{v}$ in the form $x + \mathrm{i}y$, where x and y are real. [3]

$$\frac{1+3i}{4+2i} \times \frac{4-2i}{4-2i} = \frac{4+6+12i-2i}{16+4}$$

$$= \frac{10+10i}{20}$$

$$= \frac{1}{2} + \frac{1}{2}i$$

Examiner comment

In order to score the method mark, it is not sufficient for candidates to state that they are going to multiply, they actually need to make progress with the multiplication – so, either the numerator or the denominator in the first line must be shown. The question asks for the answer in the form $x + \mathrm{i}y$, so the final answer should be given as two separate terms rather than leaving it as $\frac{1+\mathrm{i}}{2}$.

This is an example of how the new rubric on the front of the paper about showing working will be applied. Candidates need to show that they can apply the method for division by a complex number – an answer copied from a calculator will not score any marks here.

(b) State the argument of $\frac{u}{v}$. [1]

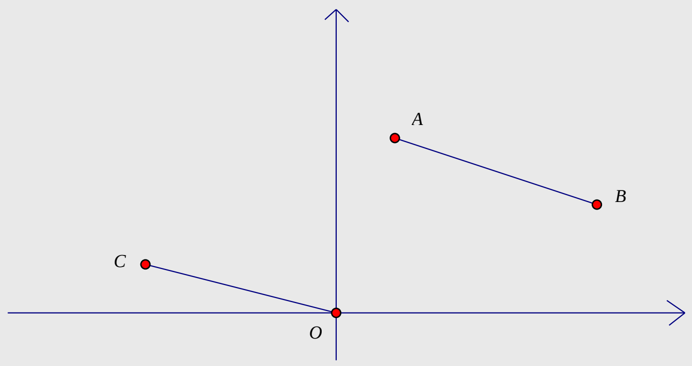
$\arg\left(\frac{1}{2} + \frac{1}{2}i\right) = \frac{\pi}{4}$ r

Examiner comment

If the answer to part **(a)** is correct then no working is expected here – it is an easily recognised angle and the answer can just be written down. If the answer to part **(a)** is not correct then the B1 mark here is still available for demonstrating a correct process for finding the angle and completing it correctly. The examiner will need to see some evidence of correct working – an incorrect answer with no working shown will not gain credit.

In an Argand diagram, with origin O, the points A, B and C represent the complex numbers u, v and $u - v$ respectively.

(c) State fully the geometrical relationship between OC and BA. [2]



OC and BA are opposite sides of a parallelogram, so equal in magnitude and parallel.

Examiner comment

It is not necessary to sketch a diagram here, but it might help if the relationship between the two lines is not clear. There is plenty of space allocated for the answer, so candidates could use some of it for a sketch. The two marks available are an indication that two distinct facts are required. Equivalent ways of stating the same properties would be acceptable, e.g. ‘same direction’ or ‘equal arguments’ in place of ‘parallel’.

(d) Show that angle $AOB = \frac{1}{4}\pi$ radians. [2]

$\angle AOB = \arg(1+3i) - \arg(4+2i)$

$= \arg\left(\frac{1+3i}{4+2i}\right) = \frac{\pi}{4}$ (from (b))

Examiner comment

This is a given result, so clear working must be shown. If the response to part (b) was correct then it is perfectly acceptable to quote that here, but only if the relevance is explained.

This is a topic that many candidates find difficult – they are not always familiar with the properties of the arguments of products and quotients of complex numbers. It is possible to answer the question by proving that triangle *OAB* is an isosceles right angled triangle, but this is not an efficient method. Candidates who attempt this method often show one of the required properties of the triangle, but not both. Another popular method is to use the cosine rule to find the required angle.

Question 7

7 (a) By first expanding $\cos(x + 45°)$, express $\cos(x+45°) - \sqrt{2}\sin x$ in the form $R\cos(x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of R correct to 4 significant figures and the value of α correct to 2 decimal places. [5]

$$\cos(x+45) - \sqrt{2}\sin x = \frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x - \sqrt{2}\sin x$$

$$= \frac{\sqrt{2}}{2}\cos x - \frac{3\sqrt{2}}{2}\sin x$$

$$\Rightarrow R^2 = \frac{2}{4} + \frac{18}{4} = 5, \qquad R = 2.236$$

$$\tan\alpha = 3, \qquad \alpha = \tan^{-1}3 = 71.57°$$

Examiner comment

For the first M1, candidates are not required to quote the general form of the expansion of $\cos(A+B)$, but they are expected to use it correctly.

Although the exact values of the coefficients are often easier to work with, so long as the final answers are correct to the required degree of accuracy, decimal forms of the coefficients would be acceptable here. The final answers need to be given to the accuracy specified in the question – this is a potential error that candidates should be aware of when they are checking through their work towards the end of the examination. The comment 'with no errors seen' in the mark scheme indicates that correct answers following errors in the working will gain no credit.

(b) Hence solve the equation

$$\cos(x + 45°) - \sqrt{2}\sin x = 2,$$

for $0° < x < 360°$. [4]

$$2.236\cos(x + 71.57) = 2$$

$$\Rightarrow \cos(x + 71.57) = \frac{2}{2.236}, \quad \cos^{-1}\left(\frac{2}{2.236}\right) = 26.56°$$

$$x + 71.57 = 360 \pm 26.56,$$

$$x = 261.9°$$

$$\text{or } x = 315.0°$$

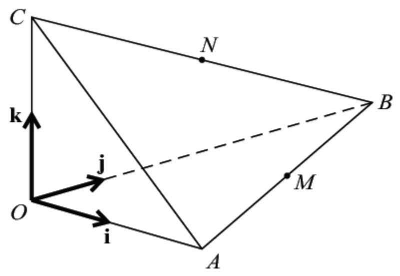
Examiner comment

The question says 'hence', so the method and values used here need to follow from the answers obtained in part **(a)**. This answer scores the B1 at the end of the second line, and the M1 at the beginning of the third line. The M1 requires a complete method to find a valid answer, so it would be scored for an answer with + or – in place of the $\pm$.

Note that one of the solutions is exactly 315° (as can easily be verified by substituting back into the original equation), so this answer would not be required to be given to one decimal place. Additional solutions outside the required range will be ignored, but the final accuracy mark will be lost if additional incorrect solutions within the range are given.

Question 8

8



In the diagram, $OABC$ is a pyramid in which $OA = 2$ units, $OB = 4$ units and $OC = 2$ units. The edge OC is vertical, the base OAB is horizontal and angle $AOB = 90°$. Unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to OA, OB and OC respectively. The midpoints of AB and BC are M and N respectively.

(a) Express the vectors $\overrightarrow{ON}$ and $\overrightarrow{CM}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. [3]

$\overrightarrow{ON} = 2\mathbf{j} + \mathbf{k}$

$\overrightarrow{CM} = \overrightarrow{CO} + \overrightarrow{OM} = -2\mathbf{k} + \mathbf{i} + 2\mathbf{j} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

Examiner comment

One of the most common issues in vector questions is candidates making transcription errors, particularly with the signs. There is only one mark available for $\overrightarrow{ON}$ so it will either be earned or not.

For $\overrightarrow{CM}$, there is a method mark available for a candidate who demonstrates a correct method of finding the required vector – candidates should clearly state what they are doing, in order to gain some credit for a correct strategy, even if there is then an error.

(b) Calculate the angle between the directions of $\overrightarrow{ON}$ and $\overrightarrow{CM}$. [3]

$\overrightarrow{ON}.\overrightarrow{CM} = 4 - 2 = 2$

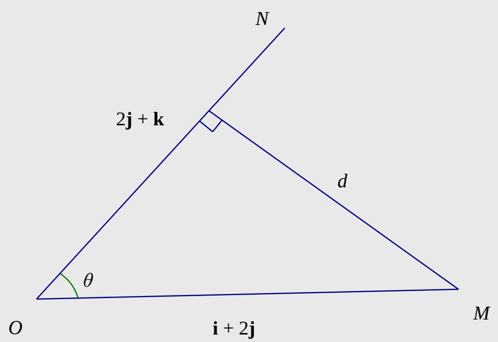
$\cos\theta = \frac{2}{\sqrt{5} \times \sqrt{9}} = \frac{2}{3\sqrt{5}}$

$\theta = 72.7°$

Examiner comment

Two of the three marks here are available for doing the right thing with the answers from part **(a)**. These marks will not be available if candidate does not show clear working. In the working here, full marks are awarded for a correct answer despite the fact that there is no explanation that $\sqrt{5} = |\overrightarrow{ON}|$ and $\sqrt{9} = |\overrightarrow{CM}|$. For the given vectors, the values of 5 and 9 are easy to calculate in your head but if either of these two values had been incorrect, then without some working to confirm what was intended, the M1 would not be awarded.

(c) Show that the length of the perpendicular from M to ON is $\frac{3}{5}\sqrt{5}$. [4]



$\cos\theta = \frac{4}{\sqrt{5}\sqrt{5}} = \frac{4}{5}$

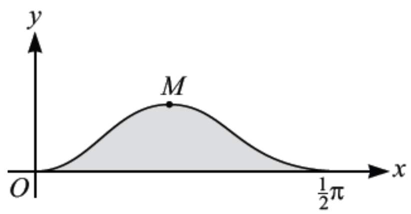
$d = |\overrightarrow{OM}|\sin\theta = \sqrt{5} \times \frac{3}{5} = \frac{3}{5}\sqrt{5}$

Examiner comment

As indicated on the mark scheme, there is more than one way to tackle this problem. This solution follows the third method. Although the question does not ask for a separate diagram, and the diagram at the top of the question is large enough for candidates to draw additional lines if required, it can be helpful to sketch a diagram with just the relevant information. There is plenty of space for the answer, so candidates can make use of it. Whichever method candidates use, they will need to give sufficient explanation of their thinking for the examiner to be able to follow what they are trying to do. The answer has been given in the question, so the examiner will be checking to make sure that the method is both valid and accurate.

Question 9

9



The diagram shows the curve $y = \sin^2 2x \cos x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$, and its maximum point M.

(a) Find the x-coordinate of M. [6]

$y = \sin^2 2x \cos x$

$\frac{dy}{dx} = 4\cos 2x \sin 2x \cos x - \sin x \sin^2 2x$

$= \sin 2x(4\cos 2x \cos x - \sin x \sin 2x)$

$= \sin 2x(4\cos^2 x \cos x - 4\sin^2 x \cos x - 2\sin^2 x \cos x)$

$= \sin 2x \cos x(4\cos^2 x - 6\sin^2 x) = 0$

$\Rightarrow \tan^2 x = \frac{2}{3}, x = 0.685$

Examiner comment

The first two marks are for the correct application of the product rule. The accuracy mark is given as soon as a correct unsimplified expression for the derivative is seen. Candidates should take care with the coefficients – the most common errors here are to overlook the negative in the derivative of $\cos x$ and to leave out the 2 when differentiating $\sin 2x$. The next stage is to equate the derivative to zero and solve the resulting equation. It is usually best to look for factors before using trig formulae to simplify terms – if candidates take out common factors as they go along then the working is often less complicated. The interval is described in radians, and candidates have used calculus with trigonometric functions, so their answer should be in radians. Remember a ‘correct’ answer with no supporting working will score no marks.

(b) Using the substitution $u = \sin x$, find the area of the shaded region bounded by the curve and the x-axis. [4]

$u = \sin x \Rightarrow \dfrac{du}{dx} = \cos x$

$x = 0,\ u = 0$
$x = \dfrac{\pi}{2},\ u = 1$

$\int_0^{\frac{\pi}{2}} \sin^2 2x \cos x \, dx = 4\int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x \cos x \, dx$

$= 4\int_0^1 u^2(1-u^2)du = 4\int_0^1 u^2 - u^4 du$

$\left[\dfrac{4}{3}u^3 - \dfrac{4}{5}u^5\right]_0^1 = \dfrac{8}{15}$

Examiner comment

The question tells candidates to use integration by substitution, so this is the method that should be seen. The substitution is a function of x, so a good starting point is to use the double angle formula to rewrite $\sin 2x$. The first two marks are for expressing the integral in terms of u – they do not need to consider the limits until the next stage. The limits for u are simple values to use, so it makes sense to complete the integral in terms of u. Some candidates prefer to convert back to an expression in terms of x and use the original limits – this often leads to more complicated working, and does not improve the chances of reaching the correct answer.

A ‘correct’ answer with no supporting working will not score any marks – the question requires candidates to demonstrate that they can work through the complete process of integration by substitution.

Question 10

10 In a chemical reaction, a compound X is formed from two compounds Y and Z.

The masses in grams of X, Y and Z present at time t seconds after the start of the reaction are x, $10-x$ and $20-x$ respectively. At any time the rate of formation of X is proportional to the product of the masses of Y and Z present at the time. When $t=0$, $x=0$ and $\frac{\mathrm{d}x}{\mathrm{d}t}=2$.

(a) Show that x and t satisfy the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.01(10-x)(20-x).$$

[1]

$$\frac{dx}{dt} = k(10-x)(20-x)$$

$$t=0,\ x=0,\ \frac{dx}{dt}=2 \Rightarrow 2 = 200k,\ k = 0.01$$

$$\Rightarrow \frac{dx}{dt} = 0.01(10-x)(20-x)$$

Examiner comment

There is only one mark available here. The question tells the candidate the form of the differential equation. They need to demonstrate that they have understood this by stating the form of the equation with the constant of proportionality and then use the remaining information to determine the value of the constant. The answer is given, so clear and detailed working is needed.

(b) Solve this differential equation and obtain an expression for x in terms of t. [9]

$$\int \frac{100}{(10-x)(20-x)}dx = \int 1 dt$$

$$\frac{100}{(10-x)(20-x)} = \frac{A}{10-x} + \frac{B}{20-x}$$

$$100 = A(20-x) + B(10-x)$$

$$x = 20,\ 100 = -10B,\ B = -10$$

$$x = 10,\ 100 = 10A,\ A = 10$$

$$\Rightarrow \int \frac{10}{10-x} - \frac{10}{20-x}dx = \int 1 dt$$

$$10\ln\left(\frac{20-x}{10-x}\right) = t + C$$

$$t = 0,\ x = 0 \Rightarrow C = 10\ln 2,\ t = 10\ln\left(\frac{20-x}{20-2x}\right)$$

$$e^{\frac{t}{10}} = \frac{20-x}{20-2x},\ x = \frac{20\left(1-e^{\frac{t}{10}}\right)}{1-2e^{\frac{t}{10}}}$$

Examiner comment

There is a choice over where to place the factor of 100 when you separate variables, so not all correct solutions will look identical. The next step is to identify the need to use partial fractions and to use an appropriate method to find values for the constants. Some candidates try to complete this task with very little evidence of any working – with some of the following marks dependent on the accuracy here, it is worth taking the time to make sure that the constants are correct. Candidates should take care with the signs and coefficients when they integrate – this is a common source of accuracy errors. All the marks up to this point are available if candidates overlook the constant of integration, but the remaining marks are dependent on it.

The question asks for x in terms of t. Candidates should apply the laws of logarithms carefully – rearrangements that break the fundamental rules will not score any marks. The question does not require the answer in a particular form, so the first expression they state of the form $x = ...$ will earn the mark if it is correct.

(c) State what happens to the value of x when t becomes large. [1]

$$x = \frac{20\left(e^{-0.1t} - 1\right)}{e^{-0.1t} - 2}$$

As t becomes very large, $e^{-0.1t} \to 0$ and $x \to \frac{20}{2} = 10$

Examiner comment

This question is about considering the limiting values of functions. Candidates should be familiar with the behaviour of e^{-t} for large values of t, and this is all they need to use (rewriting their answer to part **(b)** in terms of negative exponentials if necessary). Many candidates have the incorrect impression that they need to substitute a sequence of increasing values of t. The single mark allocated and the space available for the answer should give a hint that the expected solution will be straight forward.

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